Optimal Analytic Multiburn Trajectories

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An analytic solution is obtained for propellant-optimal transfer trajectories with the following assumptions: atmospheric effects are neglected; on burn arcs, the gravity vector is a linear function of the position vector; on coast arcs, gravity varies as an inverse square function of vehicle position; and the thrust magnitude is a constant on burn arcs. The solution technique, when applied to McCue's problem, resulted in a performance error of less than 3 fps out of 3680 fps. This solution consists of two burn arcs of approximately 200 sec and 50 sec, respectively, and an interim coast arc of approximately 6000 sec. Iteration time is of the order of 0.01 sec on the Univac 1108 computer.

Introduction

THIS analysis is concerned with the optimal analytic solution of a multiburn transfer trajectory of a vehicle between given sets of initial and final boundary conditions. The performance index of the solution is the minimization of the propellant consumed, which is equivalent to the maximization of the terminal mass. The analytic solution will be approximate, since the nonlinear set of differential equations governing this problem is assumed to have no closed-form solution. The solution, moreover, will exclude atmospheric forces and, to simplify nomenclature, will be restricted to a maximum of two burn arcs.

Previous analyses in this area are listed in the reference literature. References 1 and 2 derive the necessary and sufficient conditions for an optimal multiburn solution when the control (thrust direction and magnitude) is discontinuous (thrust magnitude only). The solutions may be classified on the basis of how and to what degree the state and costate equations are integrated.

Some approaches to the problem have been numerical. For example, Ref. 3 used a quasilinearization approach to obtain solutions for an inverse-square gravitational field. In this approach, the switching function was used to determine the switch times between the burn and coast arcs. The solutions were found to be extremely sensitive to the switch times. In Ref. 4, a two-dimensional, two-burn problem was solved using the calculus of variations with a sophisticated computer logic for determining the control variables. The method of second variations was applied in Ref. 5 with a penalty approach on the performance index and the boundary conditions. This solution technique showed strong convergence characteristics; however, an excessive amount of checkout time and computer time was required. Reference 6 used the method of perturbation functions to obtain two- and possibly three-burn solutions. In this method, the switch times were added to the control vector, with the appropriate optimality conditions and transversality conditions as additional constraint equations. Although a weighted Newton-Raphson interator worked well in the close proximity of the solution, divergence frequently occurred with a new problem. Reference 7 modified this approach to include a conjugate gradient interator (Ref. 8) and an optimal, multi-impulse solution (Ref. 9) as a starter. All problems tested with these modifications were successful, although an excessive amount of computer time was characteristic of the solution.

Other approaches to the problem have included analytical features. Reference 2, for example, discusses the solution when a constant gravitational vector is assumed; the discussion indicates that no unique optimum solution exists with this assumption. Reference 2 also derives the necessary conditions for optimal flight in an approximately inverse-square radial field. With this assumption, the costate equations were analytically integrable. In Ref. 10, the optimal multi-impulse solution was used successfully to generate optimal multiburn trajectories. The approach determined the state and costate vectors analytically by making a series expansion about the optimal impulsive trajectory which satisfies the desired boundary conditions. A polynomial approach was developed in Ref. 11, which however was dependent on the proximity to the reference path and required storage of coefficients of polynomial functions of time.

An analytic solution can also be obtained for the propellant-optimal transfer of a vehicle in a vacuum between arbitrary boundary conditions if the gravitational acceleration vector on the burn arcs is assumed to vary as a linear function of the position vector. On the coast arcs, the gravitational acceleration vector is assumed to vary as an inverse square function of position. The integrals for the costate equations on the burn arcs are elementary with this assumption; the primer vector is represented by a homogeneous, second-order, linear differential equation which is readily integrated. The solution of the state equations on the burn arcs is obtained in terms of two thrust integrals which at first appear to be intractable. By a proper change of variable and an expansion of a portion of the integrand, the two thrust integrals can be expressed as an infinite sum of known integrals (Ref. 12). The coefficient of the nth integral is determined and its value depends on whether n is odd or even. Hence, the two thrust integrals (which are evaluated recursively) can be calculated to any desired degree of accuracy. Error data on the state and costate vectors is presented in Ref. 12 for a solution which assumes a gravitational acceleration vector which varies as a linear function of the position vector.

With this approach, the state and costate vectors at a time t_i (on either a burn or coast arc) can be expressed as a function of the state and costate vectors at a previous time t_{i-1}. The coefficient matrices of these vectors are of course dependent on whether the arc is a burn or a coast. The control vector for the problem is composed of the six-component initial costate vector and the switching times for the burn and coast arcs. By introducing certain optimality and transversality conditions for a multiburn solution and using the homogeneous property of the costate equations, a sufficient number of equations are obtained in terms of the boundary conditions and the unknown components of the control vector.

Since analytic expressions exist for all the composite arcs, a differential change in the desired boundary conditions can be expressed in terms of differential changes in the control vector. Using a weighted Newton-Raphson iteration technique, convergence was readily obtained for a number of example problems.

The problems used to illustrate the solution are a time-open orbit transfer (McCue's problem, Ref. 3) and a time-open shuttle rendezvous.

Received September 16, 1971; revision received January 24, 1972. Presented at the AAS/AIAA Astrodynamics Specialists Conference 1971, Ft. Lauderdale, Fla., August 17–19, 1971, AAS Paper 71-306.

Index category: Navigation, Control, and Guidance Theory.

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Problem Definition

Given an initial state vector $S(t_1)$ defined as

$$S_1 = \begin{bmatrix} R \\ V \end{bmatrix}_1 \tag{1}$$

where R and V are, respectively, the position and velocity vectors, determine the optimal thrust vector (magnitude and direction) which maximize the terminal mass and satisfy a given set of initial and final boundary conditions. The assumptions for the problem are the following. a) Atmospheric effects are neglected. b) For simplicity of nomenclature, the solution will consist of a maximum of two burn arcs and three coast arcs. c) The arc times are free or open (the final time t_6 will be constrained for the rendezvous problem). d) The burn arcs will be affected as follows: 1) the thrust magnitude is a specified constant; 2) the gravitational acceleration vector varies as a linear function of the position vector. e) The coast arcs will be affected as follows: 1) the thrust magnitude is zero; 2) the gravitational acceleration vector varies as an inverse-square function of position.

The solution of this problem will be broken into two parts (i.e., burn arcs and coast arcs) with the optimal criteria, for joining these burn and coast arcs, treated in a later section.

Problem Solution

Burn Arcs

Assuming that the gravitational acceleration vector G_b on a burn arc can be expressed as a linear function of the position vector

$$G_b = -\omega^2 R \tag{2}$$

where the constant ω is equal to the Schuler frequency,[13] the optimal second-order differential equations representing the position and primer vectors on a burn arc are[12]

$$\ddot{R} = -\omega^2 R + (\beta c P/mp) \tag{3}$$
$$\ddot{P} = -\omega^2 P$$

where the primer vector P and its derivative $\dot{P}$ form the costate vector

$$\lambda = \begin{bmatrix} P \\ \dot{P} \end{bmatrix} \tag{4}$$

and

$$p = |P|$$

The product βc is the thrust magnitude of the engine, β is the mass flow rate, and c the effective exhaust velocity of the engine, assumed constant. Since the thrust magnitude is constant on a burn arc, mass is dissipated at the rate of

$$\dot{m} = -\beta \tag{5}$$

The differential equation for the primer vector represents, in vibration problems, the motion of a harmonic oscillator without damping and without a forcing function. The solution for P and $\dot{P}$ is represented by

$$\begin{aligned} P &= A \sin \omega t + B \cos \omega t \\ \dot{P} &= \omega (A \cos \omega t - B \sin \omega t) \end{aligned} \tag{6}$$

where

$$A = \dot{P}(0)/\omega, \quad B = P(0)$$

The transformation of the costate vector across a burn arc can thus be written as

$$\lambda(t_{i+1}) = \Psi \lambda(t_i) \tag{7}$$

where the (6×6) matrix Ψ is

$$\Psi = \begin{bmatrix} I \cos \omega t & \dfrac{I \sin \omega t}{\omega} \\ -I\omega \sin \omega t & I \cos \omega t \end{bmatrix} \tag{8}$$

and I is the (3×3) identity matrix.

The second-order differential equation for the position vector is rewritten in the following form:

$$\ddot{R} + \omega^2 R = (a_0/\mu)(P/p) \tag{9}$$

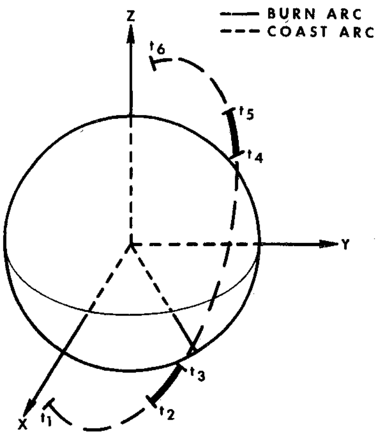

Fig. 1 Multiburn trajectory arc sequence.

where

$$a_0 = \beta c/m_0, \qquad \mu = m/m_0 = 1 + \dot{\mu} t$$

The homogeneous part of this equation has a solution

$$R = C \sin \omega t + D \cos \omega t, \quad V = \omega(C \cos \omega t - D \sin \omega t) \tag{10}$$

Using the method of variation of parameters to obtain a particular solution, the vector coefficients C and D are required to satisfy the following differential equations.

$$\begin{aligned} \dot{C} \sin \omega t + \dot{D} \cos \omega t &= 0 \\ \omega(\dot{C} \cos \omega t - \dot{D} \sin \omega t) &= (a_0/\mu)(P/p) \end{aligned} \tag{11}$$

The complete solution of Eq. (9) is obtained by solving Eq. (11) for C and D and by using these solutions in Eq. (10). The solutions for C and D are

$$C = C_1 + (a_0/\omega) I_1; \qquad D = D_1 - (a_0/\omega) I_2 \tag{12}$$

where C_1 and D_1 are constants of integration equal to

$$C_1 = [V(0)/\omega] - (a_0/\omega) I_1(0); \quad D_1 = R(0) + (a_0/\omega) I_2(0) \tag{13}$$

The integrals I_1 and I_2 are of the form

$$I_1 = \int (P/\mu p) \cos \omega t \, dt; \quad I_2 = \int (P/\mu p) \sin \omega t \, dt \tag{14}$$

One possible method for evaluating these integrals is presented in Ref. 12. The transformation of the state vector across a burn arc can be written as

$$S(t_{i+1}) = \Psi S(t_i) + \theta' \lambda(t_i) \tag{15}$$

where the matrix Ψ is given by Eq. (8) and the matrix θ' can be obtained from Ref. 12.

Coast Arcs

Assuming that the gravitational acceleration vector G_c on a coast arc varies as an inverse-square function of position.

$$G_c = -R/r^3 \tag{16}$$

where

$$r = |R|$$

(the solution is normalized on a basis of a gravitational constant of unity) the transformations for the state and costate vectors can be expressed as

$$S(t_{i+1}) = Q S(t_i); \qquad \lambda(t_{i+1}) = \Phi \lambda(t_i) \tag{17}$$

where the (6×6) matrices Q and Φ are well-known closed-form solutions (Refs. 6 and 14) for the propagation of the state and costate vectors across a coast arc.

We are now in a position to compute a completely closed-form solution across a series of coast and burn arcs. For, given an initial state vector $S(t_1)$ and a control vector consisting of the initial costate vector $\lambda(t_1)$ and a set of switch times t_i $(i = 1, 6)$ (Fig. 1), the terminal state and costate vectors $S(t_6)$, $\lambda(t_6)$ may be computed.

Boundary Condition Satisfaction

This initial solution does not, of course, satisfy some given desired boundary conditions. It is therefore necessary to make a correction to the control vector such that the dissatisfaction will decrease on the next iteration. For the purpose of elucidation, consider the mapping equations for each of the five arcs. The reason for the initial and final coast arcs will become clear when some specific boundary conditions are discussed. For each of the arcs, the state and costate equations are given in Table 1. [See Fig. 1 and Eqs. (7, 15, 17).] On arc 5, the total differential of S_6 may be expressed as

$$dS_6 = \delta S_6^- + \dot{S}_6^- dt_6 \tag{18}$$

Where δS_6^- is the contemporaneous variation of S_6 and where the superscript minus indicates a quantity evaluated at the time immediately before an event and plus will indicate evaluation at the time immediately after an event. Since on coast arcs, perturbations in the state vectors transform in the same manner as the costate vectors, Eq. (18) may be written as

$$dS_6 = \Phi_3 \delta S_5^+ + \dot{S}_6 dt_6 \tag{19}$$

Making use of the definition of a total differential to eliminate δS_5^+, Eq. (19) may be expressed as

$$dS_6 = \Phi_3 (dS_5 - \dot{S}_5^+ dt_5) + \dot{S}_6^- dt_6 \tag{20}$$

The reader should take special note of the superscript signs on the state vectors. For although the time differential of the costate vectors are continuous across a change in arc (i.e., $\dot{\lambda}_i^- = \dot{\lambda}_i^+$), this is not true for the time differential of the state vectors. The second three elements of the vectors $\dot{S}_i$ have a computable jump discontinuity across a change in arc because of a discontinuity in the thrust magnitude.

Expressing the differential change dS_5 in terms of its variation and time derivative, Eq. (20) is written as

$$dS_6 = \Phi_3 [\delta S_5^- + (\dot{S}_5^- - \dot{S}_5^+) dt_5] + \dot{S}_6^- dt_6 \tag{21}$$

From Table 1 the variation δS_5^- is

$$\delta S_5^- = \Psi_2 \delta S_4^+ + \theta_2 \delta \lambda_4 \tag{22}$$

where

$$\theta_2 \delta \lambda_4 = \delta(\theta_2' \lambda_4) = \theta_2' \delta \lambda_4 + \delta \theta_2' \lambda_4$$

which may be written as

$$\delta S_5^- = \Psi_2 (dS_4 - \dot{S}_4^+ dt_4) + \theta_2 (d\lambda_4 - \dot{\lambda}_4 dt_4) \tag{23}$$

Making use of Eq. (23) in Eq. (21), the dissatisfaction in the terminal state becomes

$$dS_6 = \Phi_3 [\Psi_2 (dS_4 - \dot{S}_4^+ dt_4) + \theta_2 (d\lambda_4 - \dot{\lambda}_4 dt_4) + (\dot{S}_5^- - \dot{S}_5^+) dt_5] + \dot{S}_6^- dt_6 \tag{24}$$

Now dS_4, similar to dS_6 in Eq. (20) may be written as

$$dS_4 = \Phi_2 (dS_3 - \dot{S}_3^+ dt_3) + \dot{S}_4^- dt_4 \tag{25}$$

The total differential of λ_4 however, will require some additional explanation. Certainly λ_4 is a function of λ_3, but it is also a function of the state at the time t_3. We can therefore write for the total differential of λ_4

$$d\lambda_4 = \frac{\partial \lambda_4}{\partial \lambda_3} \delta \lambda_3 + \frac{\partial \lambda_4}{\partial S_3} \delta S_3^+ + \dot{\lambda}_4 dt_4 \tag{26}$$

The coefficient of $\delta\lambda_3$ is, of course, the state transition matrix Φ_2. The coefficient of δS_3^+ is referred to as $d\Phi$ in Ref. 6 and it is produced as an additional output from the formulation of the state transition matrix (Ref. 14). The matrix $\partial\lambda/\partial S$ will be defined as Γ in this analysis. Thus $d\lambda_4$ may be written as

$$d\lambda_4 = \Phi_2 (d\lambda_3 - \dot{\lambda}_3 dt_3) + \Gamma_2 (dS_3 - \dot{S}_3^+ dt_3) + \dot{\lambda}_4 dt_4 \tag{27}$$

Using Eqs. (25 and 27) in Eq. (24), dS_6 becomes

$$dS_6 = \Phi_3 \{\Psi_2 [\Phi_2 (dS_3 - \dot{S}_3^+ dt_3) + (\dot{S}_4^- - \dot{S}_4^+) dt_4] + \theta_2 [\Phi_2 (d\lambda_3 - \dot{\lambda}_3 dt_3) + \Gamma_2 (dS_3 - \dot{S}_3^+ dt_3)] + (\dot{S}_5^- - \dot{S}_5^+) dt_5\} + \dot{S}_6^- dt_6 \tag{28}$$

Similar to arc 4, the differentials of the state and costate vectors on arc 2 may be expressed as

$$dS_3 = \Psi_1 (dS_2 - \dot{S}_2^+ dt_2) + \theta_1 (d\lambda_2 - \dot{\lambda}_2 dt_2) + \dot{S}_3^- dt_3 \tag{29}$$

and

$$d\lambda_3 = \Psi_1 (d\lambda_2 - \dot{\lambda}_2 dt_2) + \dot{\lambda}_3 dt_3 \tag{30}$$

Using Eqs. (29 and 30) in Eq. (28) and collecting the coefficients of the term $(dS_3 - \dot{S}_3^+ dt_3)$, the terminal boundary condition error dS_6 becomes

Table 1 State and costate equations

Arc	Type of arc	State equation	Costate equation
1	Coast	$S_2 = Q_1 S_1$	$\lambda_2 = \Phi_1 \lambda_1$
2	Burn	$S_3 = \Psi_1 S_2 + \theta_1' \lambda_2$	$\lambda_3 = \Psi_1 \lambda_2$
3	Coast	$S_4 = Q_2 S_3$	$\lambda_4 = \Phi_2 \lambda_3$
4	Burn	$S_5 = \Psi_2 S_4 + \theta_2' \lambda_4$	$\lambda_5 = \Psi_2 \lambda_4$
5	Coast	$S_6 = Q_3 S_5$	$\lambda_6 = \Phi_3 \lambda_5$

$$dS_6 = \Phi_3 \{\Lambda [\Psi_1 (dS_2 - \dot{S}_2^+ dt_2) + \theta_1 (d\lambda_2 - \dot{\lambda}_2 dt_2) + (\dot{S}_3^- - \dot{S}_3^+) dt_3] + \theta_2 \Phi_2 \Psi_1 (d\lambda_2 - \dot{\lambda}_2 dt_2) + \Psi_2 (\dot{S}_4^- - \dot{S}_4^+) dt_4 + (\dot{S}_5^- - \dot{S}_5^+) dt_5\} + \dot{S}_6^- dt_6 \tag{31}$$

where

$$\Lambda = \Psi_2 \Phi_2 + \theta_2 \Gamma_2$$

Now similar to arc 3, the differential of the state and costate vectors on arc 1 may be expressed as

$$dS_2 = \Phi_1 (dS_1 - \dot{S}_1^+ dt_1) + \dot{S}_2^- dt_2 \tag{32}$$

$$d\lambda_2 = \Phi_1 (d\lambda_1 - \dot{\lambda}_1 dt_1) + \Gamma_1 (dS_1 - \dot{S}_1^+ dt_1) + \dot{\lambda}_2 dt_2 \tag{33}$$

Consider the following boundary conditions on the initial arc. The first burn is to be initiated at an unspecified time t_2 from an optimum location (true anomaly) on the initial orbit. The state S_1 then simply defines the initial orbit and the time t_1, an arbitrary reference time. For this set of initial boundary conditions, the state S_1 and the time t_1, are considered fixed and hence their total differentials are zero.

$$\begin{aligned} dS_1 &= 0 \\ dt_1 &= 0 \end{aligned} \tag{34}$$

Using Eqs. (32, 33, and 34) in Eq. (31) and collecting coefficients, the terminal boundary condition error dS_6 is

$$dS_6 = \Phi_3 \{[\Lambda \theta_1 + \theta_2 \Phi_2 \Psi_1] \Phi_1 d\lambda_1 + \Lambda \Psi_1 (\dot{S}_2^- - \dot{S}_2^+) dt_2 + \Lambda (\dot{S}_3^- - \dot{S}_3^+) dt_3 + \Psi_2 (\dot{S}_4^- - \dot{S}_4^+) dt_4 + (\dot{S}_5^- - \dot{S}_5^+) dt_5\} + \dot{S}_6^- dt_6 \tag{35}$$

The state S_6 defines the terminal orbit. For a time-open, orbit-to-orbit transfer (initial and terminal true anomalies are free), let F_i $(i = 1, \ldots, 6)$ represent the difference between the desired terminal state and the one actually computed. It may be argued that you are actually going to a given state on the final orbit. But recall that the arrival time t_6 at this specified state is free to be determined, and that the last arc is a coast. This means that the vehicle will actually arrive on the final orbit at the time t_5 which may occur sooner or later than the time t_6. A negative terminal coast arc time equal to $(t_6 - t_5)$ is therefore admissible. The reason for the initial and final coast arcs in the formulation is now obvious. For an orbit-to-orbit, time-open transfer, the additional necessary parameters needed to determine the optimal departure and arrival true anomalies are readily available (t_2 and t_6).

Consider now the boundary conditions necessary for a rendezvous. The location of the first vehicle is specified at a time t_1, by the state vector S_1, and the location of the second vehicle at a time t_6 by the state vector S_6. If the state vectors S_1 and S_6 and the times t_1 and t_6 are fixed (the differentials are zero), then the solution which results will 1) depart from the initial orbit and arrive in the final orbit with the optimum true anomalies, 2) have switch times (t_2, t_3, t_4, t_5) which are free or open, and 3) will be required to rendezvous with the second vehicle at the time t_6. The rendezvous will actually occur at the time t_5; the two vehicles will fly in formation for the time interval $(t_6 - t_5)$, which may be negative. Hence the change that is required in Eq. (35) to change a time-open, orbit-to-orbit transfer to a rendezvous, orbit-to-orbit transfer is to simply set dt_6 equal to zero.

One further remark about boundary conditions with reference to Eq. (35): in the event that an initial coast cannot be tolerated or is fixed (booster staging, lunar ascent, etc.), the coefficient of dt_2 is simply set to zero with the time interval $(t_2 - t_1)$ reflecting the initial coast.

Additional Optimality and Transversality Conditions

$F_i\,(i=1,\ldots,6)$ represents six equations in terms of 11 possible unknowns $[\lambda_1, t_i, (i=2,3,\ldots,6)]$. The additional five constraint equations are obtained from 1) the optimality and transversality conditions which must be satisfied on an optimal multiburn trajectory, and 2) from the homogeneous property of the costate equations.

In formulating an optimal control problem for which the thrust magnitude may be discontinuous, a necessary condition is that in the interval $t_1 \leqq t \leqq t_6$, a scalar quantity known as the switch function $\$$ and defined as

$$\$ = (cp/m) - \eta - 1 \tag{36}$$

must satisfy the following conditions (Refs. 1, 2, and 6)

$$\beta = \begin{cases} \beta_{max}, \ \$ > 0 \\ 0 < \beta < \beta_{max}, \ \$ = 0 \\ 0, \ \$ < 0 \end{cases}$$

where β_{max} in the maximum allowable mass flow rate. Arcs for which the switch function is identically equal to zero over a nonzero time interval (known as singular arcs) will not be considered in this paper. The quantity η is the mass multiplier and satisfies the following differential equation:

$$\dot{\eta} = (\beta c/m^2)p \tag{37}$$

At the times t_3 and t_4 (Fig. 1), the switch function must necessarily be equal to zero. Evaluating Eq. (36) at these two times (noting that the mass m and its multiplier η are constant over the interval $t_4 - t_3$), the following condition results:

$$p_3 = p_4 \tag{38}$$

Therefore, a seventh constraint equation is

$$F_7 = p_3 - p_4 \tag{39}$$

The total differential of F_7 is

$$dF_7 = (P_3{}^T/p_3)dP_3 - (P_4{}^T/p_4)dP_4 \tag{40}$$

Now from Eq. (27), the first three elements of the differential $d\lambda_4$ may be written as

$$dP_4 = [\phi_1\, \phi_2](d\lambda_3 - \dot{\lambda}_3\, dt_3) + [\gamma_1\, \gamma_2](dS_3 - \dot{S}_3{}^+\, dt_3) + \dot{P}_4\, dt_4 \tag{41}$$

where the ϕ's and γ's are (3×3) submatrices of the (6×6) matrices Φ_2 and Γ_2. Making use of Eqs. (29, 30, 32 and 33) and the boundary conditions, Eq. (34), the differential dF_7 may be written as

$$dF_7 = \left\{\frac{P_3{}^T}{p_3}[\psi_1\, \psi_2] - \frac{P_4{}^T}{p_4}\left[[\phi_1\, \phi_2]\Psi_1 + [\gamma_1\, \gamma_2]\theta_1\right]\right\}\Phi_1\, d\lambda_1 - \frac{P_4{}^T}{p_4}[\gamma_1\, \gamma_2]\Psi_1(\dot{S}_2{}^- - \dot{S}_2{}^+)dt_2 + \left[\frac{P_3{}^T}{p_3}\dot{P}_3 - \frac{P_4{}^T}{p_4}[\gamma_1\, \gamma_2](\dot{S}_3{}^- - \dot{S}_3{}^+)\right]dt_3 - \frac{P_4{}^T}{p_4}\dot{P}_4\, dt_4 \tag{42}$$

where the (3×3) matrices ψ_1 and ψ_2 are submatrices of the (6×6) matrix Ψ_1.

Another condition which must necessarily be satisfied if the switch time t_3 is to be an optimum value, is that the Hamiltonian on the interim coast arc (constant for an autonomous system) must be zero. The Hamiltonian evaluated at the time $t_3{}^+$ is

$$F_8 = P_3{}^T\dot{V}_3{}^+ - \dot{P}_3{}^T V_3 \tag{43}$$

and may be used as an eighth constraint equation.

The total differential of F_8 is

$$dF_8 = U_1{}^T\,\delta\lambda_3 + U_2{}^T\,\delta S_3{}^+ \tag{44}$$

where

$$U_1 = \begin{bmatrix} \dot{V}_3{}^+ \\ -V_3 \end{bmatrix}, \quad U_2 = \begin{bmatrix} \dot{P}_3 \\ -\dot{P}_3 \end{bmatrix} \tag{45}$$

Making use of Eqs. (29, 30, and 32–34), the error in the Hamiltonian at the time $t_3{}^+$ is written as

$$dF_8 = (U_1{}^T\Psi_1 + U_2{}^T\theta_1)\Phi_1\, d\lambda_1 + U_2{}^T\Psi_1(\dot{S}_2{}^- - \dot{S}_2{}^+)dt_2 + U_2{}^T(\dot{S}_3{}^- - \dot{S}_3{}^+)dt_3 \tag{46}$$

Since the costate vector is represented by a homogeneous set of equations, not all of the components of the vector λ_1 are independent. To obtain an independent set we might simply chose one of the components to be set initially equal to a constant, but this approach may conceivably lead to convergence difficulties at some point in the iteration process. A better approach, with regard to the homogeneous property, is to fix the length of the initial costate vector. A ninth constraint equation is then

$$F_9 = |\lambda_1| - \text{Constant} \tag{47}$$

where the constant will be chosen to be the magnitude of the inputed costate vector such that the error on the first iterate due to this constraint will be zero. The total differential of this constraint is simply a unit vector in the direction of λ_1

$$dF_9 = (\lambda_1{}^T/|\lambda_1|)d\lambda_1 \tag{48}$$

If the initial coast arc is not fixed, i.e., the switch time t_2 is to be optimized, then the Hamiltonian on the initial coast arc must be zero. Since the Hamiltonian is constant on a coast arc, the tenth constraint equation may be expressed as

$$F_{10} = P_1{}^T\dot{V}_1{}^+ - \dot{P}_1{}^T V_1 \tag{49}$$

and its total differential is [since $dS_1 = 0$ by Eq. (34)]

$$dF_{10} = U_3{}^T\, d\lambda_1 \tag{50}$$

where

$$U_3 = \begin{bmatrix} \dot{V}_1{}^+ \\ -V_1 \end{bmatrix}$$

For the rendezvous option, the terminal time t_6 is fixed and only 10 constraint equations are required for the solution. The time-open option (t_6 is a parameter to be optimized), will require one more constraint equation than the rendezvous option. The Hamiltonian evaluated at the time $t_5{}^+$ is

$$F_{11} = P_5{}^T\dot{V}_5{}^+ - \dot{P}_5{}^T V_5 \tag{51}$$

and will be used as the eleventh constraint equation.

The total differential of F_{11} is

$$dF_{11} = U_4{}^T\,\delta\lambda_5 + U_5{}^T\,\delta S_5{}^+ \tag{52}$$

where

$$U_4 = \begin{bmatrix} \dot{V}_5{}^+ \\ -V_5 \end{bmatrix}, \quad U_5 = \begin{bmatrix} \dot{P}_5 \\ -\dot{P}_5 \end{bmatrix} \tag{53}$$

Expressing dF_{11} in terms of the control vector, there results

$$dF_{11} = [U_4{}^T\Psi_2(\Phi_2\Psi_1 + \Gamma_2\theta_2) + U_5{}^T(\Lambda\theta_1 + \theta_2\Phi_2\Psi_1)]\Phi_1\, d\lambda_1 + (U_4{}^T\Psi_2\Gamma_2 + U_5{}^T\Lambda)\Psi_1(\dot{S}_2{}^- - \dot{S}_2{}^+)dt_2 + (U_4{}^T\Psi_2\Gamma_2 + U_5{}^T\Lambda)(\dot{S}_3{}^- - \dot{S}_3{}^+)dt_3 + U_5{}^T\Psi_2(\dot{S}_4{}^- - \dot{S}_4{}^+)dt_4 + U_5{}^T(\dot{S}_5{}^- - \dot{S}_5{}^+)dt_5 \tag{54}$$

Solution Difficulties

Positive Arc Times

Some additional constraint equations may be needed to guarantee that the switch times t_3, t_4, and t_5 form a monotomically increasing set throughout the solution. Recall that the switch times are considered to be control parameters which, by definition, must be free to be chosen by the solution. The time intervals $(t_2 - t_1)$ and $(t_6 - t_5)$ will not be required to be positive since a negative arc time here will simply represent an earlier departure and arrival, respectively, from the initial and final orbits. For the remaining arc time intervals, the following additional constraints are imposed:

$$F_{i+11} = v_i(t_{i+2} - t_{i+1} - \alpha_i{}^2) \qquad i = 1, 2, 3 \tag{55}$$

where α_i's are additional control variables (real) to be determined and the v_i's are additional constant multiplies, effectively switches which obey the following rule:

$$v_i = \begin{cases} 0, \ t_{i+2} - t_{i+1} \geqq 0 \\ 1, \ t_{i+2} - t_{i+1} < 0 \end{cases}$$

The total differential of Eq. (55) is

$$dF_{i+11} = v_i(dt_{i+2} - dt_{i+1} - 2\alpha_i\, d\alpha_i) \tag{56}$$

This constraint is automatically incorporated or deleted as required by the solution.

Segmenting the Burn Arcs

In the computation of the transformations for the state and costate vectors on a burn arc, there is a possibility that the matrix θ will not converge to within the desired tolerance. The exact reason for this failure is described in Ref. 12. Simply speaking, what occurs is that either a term proportional to the burn arc time exceeds a radius of convergence or a series does not converge because of the finite word length of the computer. The solution to this problem is to break up the burn arc into a number of segments such that the convergence difficulties are not violated. Consider a burn arc of duration $(t_{j+1}-t_{j-1})$. If the arc violates the convergence criteria and is segmented into two parts, the transformation equations for the state and costate vectors are

$$S_{j+1} = \Psi_j \Psi_{j-1} S_{j-1} + (\Psi_j \theta_{j-1}' + \theta_j' \Psi_{j-1})\lambda_{j-1}$$
$$\lambda_{j+1} = \Psi_j \Psi_{j-1} \lambda_{j-1} \tag{57}$$

Since the equations remain linear in the state and costate vectors, this process may be continued until the desired tolerance is satisfied. For a burn arc segmented into two parts, the transformation matrices become

$$\Psi = \Psi_j \Psi_{j-1}; \qquad \theta' = \Psi_j \theta_{j-1}' + \theta_j' \Psi_{j-1} \tag{58}$$

This process is automatically incorporated into and deleted from the solution. An additional segment is added to the burn arc whenever a tolerance is not satisfied, and subtracted from the burn arc whenever a lesser number of segments will meet the tolerance.

Choosing the Initial Control Vector

One of the chief difficulties encountered with this solution technique (and in fact with almost all optimal two point boundary value problems) is the requirement for choosing an initial control vector and specifically with the choice of the second three elements of the vector λ_1, i.e., $\dot{P}_1$. Physically we almost always have some knowledge of the initial direction of the trust, i.e., P_1/p_1, the duration of the burn arcs, and even the length of the coast arcs. However, the vector $\dot{P}_1$ remains physically intangible. One possible choice for the direction of $\dot{P}_1$ is along the net acceleration vector, but the question still remains as to the magnitude of this vector. Another consideration which the vector $\dot{P}_1$ must, in some manner, take into account is the type of solution being considered. By this is meant that on the two burn arcs, is the orientation of the thrust vector with respect to the velocity vector: posigrade-posigrade, posigrade-retrograde, retrograde-retrograde, or retrograde-posigrade?

One possible procedure for establishing the initial costate vector is the following. From Table 1, the vector λ_4 can be expressed in terms of λ_1

$$\lambda_4 = \Phi_2 \Psi_1 \Phi_1 \lambda_1 = \Xi \lambda_1 \tag{59}$$

Six boundary conditions are needed to solve this equation. Since the direction of the thrust vector is known with some certainty at

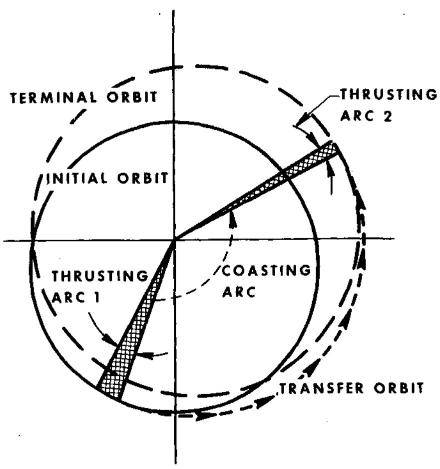

Fig. 2 McCue's problem.

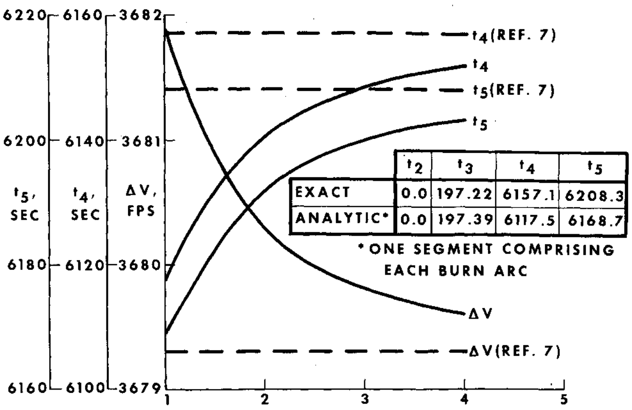

Fig. 3 Effect of segmenting the first burn arc of McCue's problem.

the times t_1 and t_4, and since the magnitude of the vector P is required to be the same at the times t_3 and t_4, an approximate solution for $\dot{P}_1$ can be obtained by solving Eq. (59) with these boundary conditions

$$\dot{P}_1 = \xi_2^{-1}(P_4 - \xi_1 P_1) \tag{60}$$

where $p_1 \simeq p_4$ and where the (3×3) matrices ξ are submatrices of the (6×6) matrix Ξ. The matrix Ξ is determined by making a preliminary pass through the solution to establish the matrices Φ_2, Ψ_1, and Φ_1 and by using the net acceleration vector for $\dot{P}_1$.

Recommendations

Choice of the Initial Control Vector

To establish the initial control vector (i.e., λ_1, t_i, $i = 1, 6$) some sort of reference solution, not requiring an initial estimate, is required. The optimal impulsive solutions are one possibility. An optimal two-impulse solution was used in Ref. 4 to obtain starting iterations for the control vector. A technique described in Ref. 13 was used to convert the impulses to finite burn arcs and Ref. 7 used a similar procedure along with an optimal multi-impulse solution technique[9] to obtain starting iteratives. A conjugate gradient algorithm[8] was also used in Ref. 7 instead of a weighted Newton-Raphson technique to achieve convergence, and with this combination, converged solutions were obtained.

For nonguidance applications, it is recommended that the convergence technique established in Ref. 7 be used to generate solutions. Using the analytic solution proposed in this paper, a considerable savings in computer time would result over using the integrated solution.

Additional Burn Arcs

Examining the differential equation for the mass multiplier, Eq. (37), it can be established that the procedure used to evaluate the thrust integrals can also be used to obtain $\eta(t)$. Knowing $\eta(t)$, $m(t)$, and $p(t)$, the time history of the switch function, Eq. (36), can be determined for the entire trajectory. The switch function can then be examined on the burn and coast arcs to determine whether an additional burn arc will improve the solution. For example, if on a coast arc the switch function becomes positive, then a burn arc inserted at this point will improve the performance of the solution. It is recommended that this addition also be made to the solution.

Numerical Results

McCue's problem (Ref. 3) was the first problem treated with this solution technique. This problem (Fig. 2) consisted of a time-open, orbit-to-orbit, two-burn transfer between coplanar ellipses, each with eccentricities of 0.2 and semilatus rectums of 5000 naut miles and 6000 naut miles, respectively. The arguments of perigee were misalined by 120°. The initial thrust-to-weight ratio of the vehicle was equal to 0.4 and the specific impulse of the engine was equal to 400 sec. The solution consisted of a near posigrade

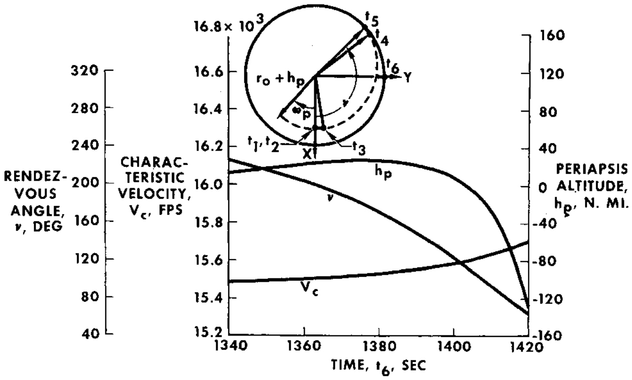

Fig. 4 Shuttle rendezvous.

burn of approximately 200 sec, a coast of approximately 6000 sec, and finally a near retrograde burn of approximately 50 sec (Fig. 3). The initial conditions for this solution were taken from Ref. 7 and the comparisons which are made are with respect to this reference. No attempt was made to determine the sensitivity of this solution to the choice of the initial control vector. The goal for this problem was to determine the sensitivity of the solution with respect to the number of segments on a burn arc and to establish the criteria for changing the number of segments.

Figure 3 illustrates the effect of segmenting the first burn arc of McCue's problem. The exact and analytic (one segment comprising each burn arc) switching times are also included in the figure. The point to be noted here is that as the number of segments increase the analytic solution, in general, converges in an exponential manner to the exact solution. The errors are quite tolerable even for one segment. For example, although the second engine ignition time t_4 had an error of aproximately 40 sec, the ΔV difference between the exact and analytic solutions amounted to less than 3 fps. With four segments, the error in the time t_4 is less than 5 sec with an accompanying ΔV difference of less than 0.3 fps. The iteration time (the time for one complete iteration) was less than 0.05 sec for this problem.

The second problem examined with this solution technique was a time-open rendezvous about the Earth between a shuttle staging point and a vehicle in a 100-naut miles circular orbit (Fig. 4). At the time $t_1 = t_2 = 0$, the first vehicle was located on the $+X$-axis with a state defined by $h = 200{,}000$ ft, $v = 11{,}000$ fps, and $\gamma = 7°$. At a time t_6, the second vehicle was located on the $+Y$-axis of a 100-naut miles circular orbit. The initial thrust-to-weight ratio of the vehicle was equal to 1.0 and the specific impulse of the engine was equal to 400 sec. The solution consisted of two posigrade burns of approximately 280 sec and 2 sec, respectively. The time t_6 was initially chosen to be equal to 1400 sec, approximately a quarter period of the circular orbit. With this time, a characteristic velocity $V_c = 15{,}588$ fps resulted with the rendezvous occurring in the second quadrant at a rendezvous angle of 123.7°, and with an altitude of periapsis of the interim coast arc $h_p = 9$ naut miles. The empirical relationship $\omega_p \simeq \nu - 174.0°$ was noted for the argument of periapsis of the interim coast arc and the rendezvous angle. The true anomalies on this interim coast arc at the times t_3 and t_5 were 63.1° and 173.9°, respectively. As the time t_6 is increased above 1400 sec, the characteristic velocity increases slightly and the radius of periapsis and the rendezvous angle tend toward zero. Thus, there is an upper bound on the time t_6 at which the two burns would merge into one burn and the rendezvous would occur without an interim coast. As the time t_6 is decreased below 1400 sec, the characteristic velocity decreases, the altitude of periapsis goes through a maximum and the rendezvous angle increases to 2π. As the rendezvous angle becomes larger than 174° plus the first burn angle ($\simeq 13°$), the vehicle passes through periapsis on the interim coast arc. Thus, there is a lower bound on the time t_6 determined by the minimum allowable periapsis altitude.

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